Due Fri

Warm up/review

$$\begin{bmatrix}1\\0\\0\end{bmatrix}x_1+\begin{bmatrix}-2\\0\\0\end{bmatrix}x_2+\begin{bmatrix}0\\1\\0\end{bmatrix}x_3+\begin{bmatrix}3\\1\\0\end{bmatrix}x_4+\begin{bmatrix}0\\0\\1\end{bmatrix}x_5$$

(6 pts) The augmented matrix for a linear system of equations has been reduced to reduced row echelon form. Express the solution set as a linear combination of column vectors that contain only numerical entries.

free → Parameters

$$\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \\ 1 & -2 & 0 & 3 & 0 & 4 \\ 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 6 \end{array}$$

leading variables

$$x_1 - 2x_2 + 3x_4 = 4$$
$$x_3 + x_4 = -1$$
$$x_5 = 6$$

$$A\vec{x} = \vec{b}$$

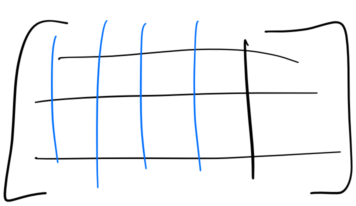
has solution $\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}$

Let $s = x_2$, $t = x_4$

$$x_3 = -x_4 - 1$$
$$= -t - 1$$

$$\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix} = \begin{bmatrix}2s-3t+4\\s\\-t-1\\t\\6\end{bmatrix}$$

$$\boxed{= \begin{bmatrix}2\\1\\0\\0\\0\end{bmatrix}s + \begin{bmatrix}-3\\0\\-1\\1\\0\end{bmatrix}t + \begin{bmatrix}4\\0\\-1\\0\\6\end{bmatrix}}$$



Augmented matrix w/ more columns than rows that reduces to a dependent system

Considering polynomial interpolation

$$P(x) = \underline{a_3}\, x^3 + \underline{a_2}\, x^2 + \underline{a_1}\, x + \underline{a_0}$$

(x_1, y_1) 1
(x_2, y_2) -1
(x_3, y_3) 2
(x_4, y_4) 3

$$\left[\begin{array}{cccc|c} x_1^3 & x_1^2 & x_1 & 1 & y_1 \\ 1 & & & & \\ -1 & & & & \\ 8 & & & & \\ 27 & & & & \end{array}\right]$$

and so on

$A\vec{x} = \vec{b}$

$$\begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix}$$

$$P(x) = a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^3$$

$(2, 3)$: $a_0 + 2a_1 + 4a_2 + 8a_3 = 3$

$(5, 1)$: $a_0 + 5a_1 + 25a_2 + 125a_3 = 1$

$(-1, 7)$: $a_0 - a_1 + a_2 - a_3 = 7$

$(4, 2)$: $a_0 + 4a_1 + 16a_2 + 64a_3 = 2$

$(0, 0)$

$$\left[\begin{array}{cccc|c} 1 & 2 & 4 & 8 & 3 \\ 1 & 5 & 25 & 125 & 1 \\ 1 & -1 & 1 & -1 & 7 \\ 1 & 4 & 16 & 64 & 2 \end{array}\right]$$

$A\vec{x} = \vec{b}$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

2.1 – Determinants by Cofactor Expansion

Definition 1: If A is a square matrix, then the **minor of entry a_{ij}** is denoted by M_{ij} and is defined to be the determinant of the submatrix that remains after the ith row and jth column are deleted from A. The number $(-1)^{i+j}M_{ij}$ is denoted by C_{ij} and is called the **cofactor of entry a_{ij}**.

a_{11} a_{12} a_{13}

+ or −: $(-1)^{i+j}$

Ex: Find all the minors and cofactors of $A = \begin{bmatrix} 2^{+} & 3^{-} & -7^{+} \\ 4^{-} & -2^{+} & -9^{-} \\ 2^{+} & 5^{-} & 6^{+} \end{bmatrix}$.

Arrow technique for 2x2 determinants

$M_{11} = \begin{vmatrix} -2 & -9 \\ 5 & 6 \end{vmatrix}$ $\quad M_{12} = \begin{vmatrix} 4 & -9 \\ 2 & 6 \end{vmatrix}$ $\quad M_{13} = \begin{vmatrix} 4 & -2 \\ 2 & 5 \end{vmatrix}$

$-2(6) - 5(-9)$ $\quad M_{12} = 42$ $\quad M_{13} = 24$

$M_{11} = 33$

$C_{11} = 33$ $\quad C_{12} = -42$ $\quad C_{13} = 24$

$M_{21} = \begin{vmatrix} 3 & -7 \\ 5 & 6 \end{vmatrix}$ $\quad M_{22} = \begin{vmatrix} 2 & -7 \\ 2 & 6 \end{vmatrix}$ $\quad M_{23} = \begin{vmatrix} 2 & 3 \\ 2 & 5 \end{vmatrix}$

$M_{21} = 53$ $\quad M_{22} = 26$ $\quad M_{23} = 4$

$C_{21} = -53$ $\quad C_{22} = 26$ $\quad C_{23} = -4$

$M_{31} = \begin{vmatrix} 3 & -7 \\ -2 & -9 \end{vmatrix}$ $\quad M_{32} = \begin{vmatrix} 2 & -7 \\ 4 & -9 \end{vmatrix}$ $\quad M_{33} = \begin{vmatrix} 2 & 3 \\ 4 & -2 \end{vmatrix}$

$M_{31} = -41$ $\quad M_{32} = 10$ $\quad M_{33} = -16$

$C_{31} = -41$ $\quad C_{32} = -10$ $\quad C_{33} = -16$

Definition 2: If A is an $n \times n$ matrix, then the number obtained by multiplying the entries in any row or column of A by the corresponding cofactors and adding the resulting products is called the **determinant of A**, and the sums themselves are called **cofactor expansions of A**. That is,

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

(cofactor expansion along the jth column)

$$= a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

(cofactor expansion along the ith row)

determinant notation

Ex: Compute the determinant of the matrix A above.

Matrix: $\begin{bmatrix} 2 & 3 & -7 \\ 4 & -2 & -9 \\ 2 & 5 & 6 \end{bmatrix}$ determinant: $\begin{array}{ccc} + & - & + \end{array}$ $\begin{vmatrix} 2 & 3 & -7 \\ 4 & -2 & -9 \\ 2 & 5 & 6 \end{vmatrix}$

$2\begin{vmatrix} -2 & -9 \\ 5 & 6 \end{vmatrix} - 3\begin{vmatrix} 4 & -9 \\ 2 & 6 \end{vmatrix} - 7\begin{vmatrix} 4 & -2 \\ 2 & 5 \end{vmatrix}$

$\det(A) = |A| = +2(33) - 3(42) + (-7)(24)$

$= -228$

11. Use the arrow technique of Figure 2.1.1 to evaluate the determinant.

$\begin{vmatrix} -2 & 1 & 4 \\ 3 & 5 & -7 \\ 1 & 6 & 2 \end{vmatrix} \begin{matrix} -2 & 1 \\ 3 & 5 \\ 1 & 6 \end{matrix}$

$20 + 84 + 6 = 110$

$-20 - 7 + 72 = 45$

2×2: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$45 - 110 = \boxed{-65}$

* only ~~for~~ 3×3 *

$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} \cdots$

$= aei - afh$

$-afh$

$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \begin{matrix} a & b \\ d & e \\ g & h \end{matrix}$

aei

Theorem 2.1.1 If A is an $n \times n$ matrix, then regardless of which row or column of A is chosen, the number obtained by multiplying the entries in that row or column by the corresponding cofactors and adding the resulting products is always the same.

lambda

16. Find all values of λ for which $\det(A) = 0$.

$$A = \begin{bmatrix} \lambda - 4 & 0 & 0 \\ 0 & \lambda & 2 \\ 0 & 3 & \lambda - 1 \end{bmatrix}$$

1st row

$$\begin{vmatrix} \lambda-4 & 0 & 0 \\ 0 & \lambda & 2 \\ 0 & 3 & \lambda-1 \end{vmatrix} = (\lambda-4)\begin{vmatrix} \lambda & 2 \\ 3 & \lambda-1 \end{vmatrix} - 0 + 0 = 0$$

$$\Rightarrow (\lambda-4)(\lambda(\lambda-1) - 6) = 0$$

$$(\lambda-4)(\lambda^2 - \lambda - 6) = 0$$

$$(\lambda-4)(\lambda-3)(\lambda+2) = 0 \Rightarrow \boxed{\lambda = -2, 3, 4}$$

Note: Let $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ $\quad 2A = \begin{bmatrix} 4 & 6 \\ 8 & 10 \end{bmatrix}$

$\det(A) = -2$ $\quad 2\det(A) = -4$ $\quad \det(2A) = -8$

So $\det(2A) \neq 2 \det A$

That is, $2\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} \neq \det\left(2\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\right)$

Theorem 2.1.2 If A is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then $\det(A)$ is the product of the entries on the main diagonal of the matrix; that is, $\det(A) = a_{11}a_{22} \cdots a_{nn}$.

31. Evaluate the determinant of the given matrix by inspection.

$$A = \begin{bmatrix} 1 & 2 & 7 & -3 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$\det(A) = 1(1)(2)(3) = 6$